Size and Logic

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Abstract

We show how to develop a multitude of rules of nonmonotonic logic from very simple and natural notions of size, using them as building blocks.

Contents

1 Introduction

1.1 Context

The study of modal and temporal logic and the study of substructural logic went for many years along the following lines: on the one hand we had syntactic proof theoretic systems, mainly Gentzen or Hilbert systems and on the other hand we had semantical interpretations, mainly possible worlds or algebraic structures, and the community very thoroughly analysed properties of one against matching properties of the other. The success of such depended on the correct identification of the correct features in the semantics. In the case of nonmonotonic logic there is the syntactical consequence relation on the one hand and the preferential ordering on the other but the research is not yet in a similar comprehensive stage as in the other areas. In this paper we use the important semantical feature of size to display a detailed matching between syntactical and semantical conditions for non monotonic systems.

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We show how one can develop a multitude of rules for nonmonotonic logics from a very small set of principles about reasoning with size. The notion of size gives an algebraic semantics to nonmonotonic logics, in the sense that α implies β iff the set of cases where $\alpha \wedge \neg\beta$ holds is a small subset of all $\alpha-$cases. In a similar way, e.g. Heyting algebras are an algebraic semantics for intuitionistic logic.

In our understanding, algebraic semantics describe the abstract properties corresponding model sets have. Structural semantics, on the other hand, give intuitive concepts like accessibility or preference, from which properties of model sets, and thus algebraic semantics, originate.

Varying properties of structural semantics (e.g. transitivity, etc.) result in varying properties of algebraic semantics, and thus of logical rules. We consider operations directly on the algebraic semantics and their logical consequences, and we see that simple manipulations of the size concept result in most rules of nonmonotonic logics. Even more, we show how to generate new rules from those manipulations. The result is one big table, which, in a much more modest scale, can be seen as a "periodic table" of the "elements" of nonmonotonic logic. Some simple underlying principles allow to generate them all.

Historical remarks: The first time that abstract size was related to nonmonotonic logics was, to our knowledge, in the second author's [Sch90] and [Sch95-1], and, independently, in [BB94]. More detailed remarks can e.g. be found in [GS08c]. But, again to our knowledge, connections are elaborated systematically and in fine detail here for the first time.

1.2 Overview

The main part of this paper is the big table in Section 2.6 (page 4). It shows connections and how to develop a multitude of logical rules known from nonmonotonic logics by combining a small number of principles about size. We use them as building blocks to construct the rules from.

These principles are some basic and very natural postulates, (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, and a continuum of power of the notion of "small", or, dually, "big", from $(1 * s)$ to $(< \omega * s)$. From these, we can develop the rest except, essentially, Rational Monotony, and thus an infinity of different rules.

This is a conceptual paper, and it does not contain any more difficult formal results. The interest lies, in our opinion, in the simplicity, paucity, and naturalness of the basic building blocks. We hope that this schema brings more and deeper order into the rich fauna of nonmonotonic and related logics.

2 Main table

LABEL: Section Table

2.1 Notation

(1) $\mathcal{P}(X)$ is the power set of X, $\subseteq$ is the subset relation, $\subset$ the strict part of $\subseteq$, i.e. $A \subset B$ iff $A \subseteq B$ and $A \neq B$. The operators $\wedge$, $\neg$, $\vee$, $\rightarrow$ and $\vdash$ have their usual, classical interpretation.

(2) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ and $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ are dual abstract notions of size, $\mathcal{I}(X)$ is the set of "small" subsets of X, $\mathcal{F}(X)$ the set of "big" subsets of X. They are dual in the sense that $A \in \mathcal{I}(X) \Leftrightarrow X - A \in \mathcal{F}(X)$. " $\mathcal{I}$ " evokes "ideal", " $\mathcal{F}$ " evokes "filter" though the full strength of both is reached only in $(< \omega * s)$. "s" evokes "small", and " $(x * s)$ " stands for " x small sets together are still not everything".

(3) If $A \subseteq X$ is neither in $\mathcal{I}(X)$, nor in $\mathcal{F}(X)$, we say it has medium size, and we define $\mathcal{M}(X) := \mathcal{P}(X) - (\mathcal{I}(X) \cup \mathcal{F}(X))$. $\mathcal{M}^+(X) := \mathcal{P}(X) - \mathcal{I}(X)$ is the set of subsets which are not small.

(4) $\nabla x \phi$ is a generalized first order quantifier, it is read "almost all x have property ϕ ". $\nabla x(\phi : \psi)$ is the relativized version, read: "almost all x with property ϕ have also property ψ ". To keep the table simple, we write mostly only the non-relativized versions. Formally, we have $\nabla x \phi :\Leftrightarrow \{x : \phi(x)\} \in \mathcal{F}(U)$ where U is the universe, and $\nabla x(\phi : \psi) :\Leftrightarrow \{x : (\phi \wedge \psi)(x)\} \in \mathcal{F}(\{x : \phi(x)\})$. Soundness and completeness results on ∇ can be found in [Sch95-1].

(5) Analogously, for propositional logic, we define:

$\alpha \mid\sim \beta :\Leftrightarrow M(\alpha \wedge \beta) \in \mathcal{F}(M(\alpha))$,

where $M(\phi)$ is the set of models of ϕ.

(6) In preferential structures, $\mu(X) \subseteq X$ is the set of minimal elements of X. This generates a principal filter by $\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$. Corresponding properties about μ are not listed systematically.

(7) The usual rules (AND) etc. are named here (AND_ω), as they are in a natural ascending line of similar rules, based on strengthening of the filter/ideal properties.

2.2 The groupes of rules

The rules are divided into 5 groups:

(1) (Opt), which says that "All" is optimal - i.e. when there are no exceptions, then a soft rule $\mid\sim$ holds.

(2) 3 monotony rules:

(2.1) (iM) is inner monotony, a subset of a small set is small, (2.2) $(eM\mathcal{I})$ external monotony for ideals: enlarging the base set keeps small sets small,

(2.3) $(eM\mathcal{F})$ external monotony for filters: a big subset stays big when the base set shrinks.

These three rules are very natural if "size" is anything coherent over change of base sets. In particular, they can be seen as weakening.

(3) $(\approx)$ keeps proportions, it is here mainly to point the possibility out.

(4) a group of rules $x * s$, which say how many small sets will not yet add to the base set.

(5) Rational monotony, which can best be understood as robustness of $\mathcal{M}^+$, see $(\mathcal{M}^{++})(3)$.

2.2.1 Regularities

(1) The group of rules $(x * s)$ use ascending strength of $\mathcal{I}/\mathcal{F}$.

(2) The column $(\mathcal{M}^+)$ contains interesting algebraic properties. In particular, they show a strengthening from $(3 * s)$ up to Rationality. They are not necessarily equivalent to the corresponding (I_x) rules, not even in the presence of the basic rules. The examples show that care has to be taken when considering the different variants.

(3) Adding the somewhat superflous (CM_2), we have increasing cautious monotony from (wCM) to full (CM_ω).

(4) We have increasing "or" from (wOR) to full (OR_ω).

(5) The line $(2 * s)$ is only there because there seems to be no $(\mathcal{M}_2^+)$, otherwise we could begin $(n * s)$ at $n = 2$.

2.3 Direct correspondences

Several correspondences are trivial and are mentioned now. Somewhat less obvious (in)dependencies are given in Section 3 (page 6) . Finally, the connections with the $\mu-$rules are given in Section 4 (page 11). In those rules, (I_ω) is implicit, as they are about principal filters. Still, the $\mu-$rules are written in the main table in their intuitively adequate place.

(1) The columns "Ideal" and "Filter" are mutually dual, when both entries are defined.

(2) The correspondence between the ideal/filter column and the $\nabla-$column is obvious, the latter is added only for completeness' sake, and to point out the trivial translation to first order logic.

(3) The ideal/filter and the AND-column correspond directly.

(4) We can construct logical rules from the $\mathcal{M}^+ - column$ by direct correspondence, e.g. for $(\mathcal{M}_\omega^+)$, (1):
Set $Y := M(\gamma)$, $X := M(\gamma \wedge \beta)$, $A := M(\gamma \wedge \beta \wedge \alpha)$.

- $X \in \mathcal{M}^+(Y)$ will become $\gamma \not\mid\!\sim \neg\beta$
- $A \in \mathcal{F}(X)$ will become $\gamma \wedge \beta \mid\!\sim \alpha$
- $A \in \mathcal{M}^+(Y)$ will become $\gamma \not\mid\!\sim \neg(\alpha \wedge \beta)$.

so we obtain $\gamma \not\mid\!\sim \neg\beta, \gamma \wedge \beta \mid\!\sim \alpha \Rightarrow \gamma \not\mid\!\sim \neg(\alpha \wedge \beta)$.
We did not want to make the table too complicated, so such rules are not listed in the table.

(5) Various direct correspondences:

- In the line (Opt), the filter/ideal entry corresponds to (SC),
- in the line (iM), the filter/ideal entry corresponds to (RW),
- in the line $(eM\mathcal{I})$, the ideal entry corresponds to (PR') and (wOR),
- in the line $(eM\mathcal{F})$, the filter entry corresponds to (wCM),
- in the line $(\approx)$, the filter/ideal entry corresponds to $(disjOR)$,
- in the line $(1 * s)$, the filter/ideal entry corresponds to (CP),
- in the line $(2 * s)$, the filter/ideal entry corresponds to $(CM_2) = (OR_2)$.

(6) Note that one can, e.g., write (AND_2) in two flavours:

- $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow \alpha \not\vdash \neg\beta \vee \neg\beta'$, or
- $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$

(which is $(CM_2) = (OR_2)$.)
For reasons of simplicity, we mention only one.

2.4 Rational Monotony

$(RatM)$ does not fit into adding small sets. We have exhausted the combination of small sets by $(< \omega * s)$, unless we go to languages with infinitary formulas.

The next idea would be to add medium size sets. But, by definition, $2 * medium$ can be all. Adding small and medium sets would not help either: Suppose we have a rule $medium + n * small \neq all$. Taking the complement of the first medium set, which is again medium, we have the rule $2 * n * small \neq all$. So we do not see any meaningful new internal rule. i.e. without changing the base set.

Probably, $(RatM)$ has more to do with independence: by default, all "normalities" are independent, and intersecting with another formula preserves normality.

2.5 Summary

We can obtain all rules except $(RatM)$ and $(\approx)$ from (Opt), the monotony rules - (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ -, and $(x * s)$ with increasing x.

2.6 Main table

LABEL: Section Main-Table

	"Ideal"	:	"Filter"	$\mathcal{M}^+$	∇	various rules	AND	OR	Caut./Rat.Mon.
Optimal proportion									
(Opt)	$\emptyset \in \mathcal{I}(X)$	:	$X \in \mathcal{F}(X)$		$\forall x\phi \to \nabla x\phi$	(SC) $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$			
Monotony (Improving proportions). (iM): internal monotony, $(eM\mathcal{I})$: external monotony for ideals, $(eM\mathcal{F})$: external monotony for filters									
(iM)	$A \subseteq B \in \mathcal{I}(X) \Rightarrow$ $A \in \mathcal{I}(X)$	:	$A \in \mathcal{F}(X)$, $A \subseteq B \subseteq X$ $\Rightarrow B \in \mathcal{F}(X)$		$\nabla x\phi \wedge \forall x(\phi \to \phi')$ $\to \nabla x\phi'$	(RW) $\alpha \mid\!\sim \beta, \beta \vdash \beta' \Rightarrow$ $\alpha \mid\!\sim \beta'$			
$(eM\mathcal{I})$	$X \subseteq Y \Rightarrow$ $\mathcal{I}(X) \subseteq \mathcal{I}(Y)$	:			$\nabla x(\phi : \psi) \wedge$ $\forall x(\phi' \to \psi) \to$ $\nabla x(\phi \vee \phi' : \psi)$	(PR') $\alpha \mid\!\sim \beta, \alpha \vdash \alpha'$, $\alpha' \wedge \neg\alpha \vdash \beta \Rightarrow$ $\alpha' \mid\!\sim \beta$ (μPR) $X \subseteq Y \Rightarrow$ $\mu(Y) \cap X \subseteq \mu(X)$		(wOR) $\alpha \mid\!\sim \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$ (μwOR) $\mu(X \cup Y) \subseteq \mu(X) \cup Y$	
$(eM\mathcal{F})$		:	$X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq \mathcal{F}(X)$		$\nabla x(\phi : \psi) \wedge$ $\forall x(\psi \wedge \phi \to \phi') \to$ $\nabla x(\phi \wedge \phi' : \psi)$				(wCM) $\alpha \mid\!\sim \beta, \alpha' \vdash \alpha$, $\alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\!\sim \beta$
Keeping proportions									
$(\approx)$	$(\mathcal{I} \cup disj)$ $A \in \mathcal{I}(X)$, $B \in \mathcal{I}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{I}(X \cup Y)$	:	$(\mathcal{F} \cup disj)$ $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{F}(X \cup Y)$		$\nabla x(\phi : \psi) \wedge$ $\nabla x(\phi' : \psi) \wedge$ $\neg\exists x(\phi \wedge \phi') \to$ $\nabla x(\phi \vee \phi' : \psi)$			$(disjOR)$ $\phi \mid\!\sim \psi$, $\phi' \mid\!\sim \psi'$ $\phi \vdash \neg\phi'$, $\Rightarrow$ $\phi \vee \phi' \mid\!\sim \psi \vee \psi'$ $(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
Robustness of proportions: $n * small \neq All$									
$(1 * s)$	$(\mathcal{I}_1)$ $X \notin \mathcal{I}(X)$	:	$(\mathcal{F}_1)$ $\emptyset \notin \mathcal{F}(X)$		(∇_1) $\nabla x\phi \to \exists x\phi$	(CP) $\phi \mid\!\sim \bot \Rightarrow \phi \vdash \bot$	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\vdash \neg\beta$		
$(2 * s)$	$(\mathcal{I}_2)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \neq X$	:	$(\mathcal{F}_2)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \neq \emptyset$		(∇_2) $\nabla x\phi \wedge \nabla x\psi$ $\to \exists x(\phi \wedge \psi)$		(AND_2) $\alpha \mid\!\sim \beta$, $\alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \not\vdash \neg\beta \vee \neg\beta'$	(OR_2) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$	(CM_2) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$
$(n * s)$ $(n \geq 3)$	$(\mathcal{I}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cup . \cup A_n \neq X$	:	$(\mathcal{F}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cap . \cap A_n \neq \emptyset$	$(\mathcal{M}^+_n)$ $X_1 \in \mathcal{F}(X_2), .,$ $X_{n-1} \in \mathcal{F}(X_n) \Rightarrow$ $X_1 \in \mathcal{M}^+(X_n)$	(∇_n) $\nabla x\phi_1 \wedge . \wedge \nabla x\phi_n$ $\to$ $\exists x(\phi_1 \wedge . \wedge \phi_n)$		(AND_n) $\alpha \mid\!\sim \beta_1, ., \alpha \mid\!\sim \beta_n \Rightarrow$ $\alpha \not\vdash \neg\beta_1 \vee . \vee \neg\beta_n$	(OR_n) $\alpha_1 \mid\!\sim \beta, ., \alpha_{n-1} \mid\!\sim \beta$ $\Rightarrow$ $\alpha_1 \vee . \vee \alpha_{n-1} \not\mid\!\sim \neg\beta$	(CM_n) $\alpha \mid\!\sim \beta_1, ., \alpha \mid\!\sim \beta_{n-1}$ $\Rightarrow$ $\alpha \wedge \beta_1 \wedge . \wedge \beta_{n-2} \not\mid\!\sim \neg\beta_{n-1}$
$(< \omega * s)$	$(\mathcal{I}_\omega)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \in \mathcal{I}(X)$	:	$(\mathcal{F}_\omega)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \in \mathcal{F}(X)$	$(\mathcal{M}^+_\omega)$ (1) $A \in \mathcal{F}(X)$, $X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (2) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (3) $A \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{F}(Y)$ (4) $A, B \in \mathcal{I}(X) \Rightarrow$ $A - B \in \mathcal{I}(X - B)$	(∇_ω) $\nabla x\phi \wedge \nabla x\psi \to$ $\nabla x(\phi \wedge \psi)$		(AND_ω) $\alpha \mid\!\sim \beta$, $\alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \mid\!\sim \beta \wedge \beta'$	(OR_ω) $\alpha \mid\!\sim \beta$, $\alpha' \mid\!\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$ (μOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	(CM_ω) $\alpha \mid\!\sim \beta$, $\alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\!\sim \beta'$ (μCM) $\mu(X) \subseteq Y \subseteq X \Rightarrow$ $\mu(Y) \subseteq \mu(X)$
Robustness of $\mathcal{M}^+$									
$(\mathcal{M}^{++})$		:		$(\mathcal{M}^{++})$ (1) $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{I}(X - B)$ (2) $A \in \mathcal{F}(X)$, $B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{F}(X - B)$ (3) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$					$(RatM)$ $\phi \mid\!\sim \psi, \phi \not\mid\!\sim \neg\psi' \Rightarrow$ $\phi \wedge \psi' \mid\!\sim \psi$ $(\mu RatM)$ $X \subseteq Y$, $X \cap \mu(Y) \neq \emptyset \Rightarrow$ $\mu(X) \subseteq \mu(Y) \cap X$

3 Coherent systems

LABEL: Section Coherent-Systems

3.1 Definition and basic facts

Note that whenever we work with model sets, the rule
(LLE), left logical equivalence, $\vdash \alpha \leftrightarrow \alpha' \Rightarrow (\alpha \mid\!\sim \beta \Leftrightarrow \alpha' \mid\!\sim \beta)$
will hold. We will not mention this any further.

Definition 3.1

(+++ Orig. No.: Definition CoherentSystem +++)

LABEL: Definition CoherentSystem

A coherent system of sizes, $\mathcal{CS}$, consists of a universe U, $\emptyset \notin \mathcal{Y} \subseteq \mathcal{P}(U)$, and for all $X \in \mathcal{Y}$ a system $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ (dually $\mathcal{F}(X)$, i.e. $A \in \mathcal{F}(X) \Leftrightarrow X - A \in \mathcal{I}(X)$). $\mathcal{Y}$ may satisfy certain closure properties like closure under $\cup$, $\cap$, complementation, etc. We will mention this when needed, and not obvious.

We say that $\mathcal{CS}$ satisfies a certain property iff all $X, Y \in \mathcal{Y}$ satisfy this property.

$\mathcal{CS}$ is called basic or level 1 iff it satisfies (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, $(1 * s)$.

$\mathcal{CS}$ is level x iff it satisfies (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, $(x * s)$.

Fact 3.1

(+++ Orig. No.: Fact 1-element +++)

LABEL: Fact 1-element

Note that, if for any Y $\mathcal{I}(Y)$ consists only of subsets of at most 1 element, then $(eM\mathcal{F})$ is trivially satisfied for Y and its subsets by (Opt). $\square$

Fact 3.2

(+++ Orig. No.: Fact Not-2*s +++)

LABEL: Fact Not-2*s

Let a $\mathcal{CS}$ be given s.t. $\mathcal{Y} = \mathcal{P}(U)$. If $X \in \mathcal{Y}$ satisfies $(\mathcal{M}^{++})$, but not $(< \omega * s)$, then there is $Y \in \mathcal{Y}$ which does not satisfy $(2 * s)$.

Proof

(+++ Orig.: Proof +++)

We work with version (1) of $(\mathcal{M}^{++})$, we will see in Fact 3.9 (page 10) that all three versions are equivalent.

As X does not satisfy $(< \omega * s)$, there are $A, B \in \mathcal{I}(X)$ s.t. $A \cup B \in \mathcal{M}^+(X)$. $A \in \mathcal{I}(X)$, $A \cup B \in \mathcal{M}^+(X) \Rightarrow X - (A \cup B) \notin \mathcal{F}(X)$, so by $(\mathcal{M}^{++})(1)$ $A = A - (X - (A \cup B)) \in \mathcal{I}(X - (X - (A \cup B))) = \mathcal{I}(A \cup B)$. Likewise $B \in \mathcal{I}(A \cup B)$, so $(2 * s)$ does not hold for $A \cup B$. $\square$

Fact 3.3

(+++ Orig. No.: Fact Independence-eM +++)

LABEL: Fact Independence-eM

$(eM\mathcal{I})$ and $(eM\mathcal{F})$ are formally independent, though intuitively equivalent.

Proof

(+++ Orig.: Proof +++)

Let $U := \{x, y, z\}$, $X := \{x, z\}$, $\mathcal{Y} := \mathcal{P}(U) - \{\emptyset\}$

(1) Let $\mathcal{F}(U) := \{A \subseteq U : z \in A\}$, $\mathcal{F}(Y) = \{Y\}$ for all $Y \subset U$. (Opt), (iM) hold, $(eM\mathcal{I})$ holds trivially, so does $(< \omega * s)$, but $(eM\mathcal{F})$ fails for U and X.

(2) Let $\mathcal{F}(X) := \{\{z\}, X\}$, $\mathcal{F}(Y) := \{Y\}$ for all $Y \subseteq U$, $Y \neq X$. (Opt), (iM), $(< \omega * s)$ hold trivially, $(eM\mathcal{F})$ holds by Fact 3.1 (page 6). $(eM\mathcal{I})$ fails, as $\{x\} \in \mathcal{I}(X)$, but $\{x\} \notin \mathcal{I}(U)$.

□

Fact 3.4

(+++ Orig. No.: Fact Level-n-n+1 +++)

LABEL: Fact Level-n-n+1

A level n system is strictly weaker than a level $n+1$ system.

Proof

(+++ Orig.: Proof +++)

Consider $U := \{1, \ldots, n+1\}$, $\mathcal{Y} := \mathcal{P}(U) - \{\emptyset\}$. Let $\mathcal{I}(U) := \{\emptyset\} \cup \{\{x\} : x \in U\}$, $\mathcal{I}(X) := \{\emptyset\}$ for $X \neq U$. (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ hold trivially. $(n * s)$ holds trivially for $X \neq U$, but also for U. $((n+1) * s)$ does not hold for U. □

Remark 3.5

(+++ Orig. No.: Remark Infin +++)

LABEL: Remark Infin

Note that our schemata allow us to generate infintely many new rules, here is an example:

Start with A, add $s_{1,1}$, $s_{1,2}$ two sets small in $A \cup s_{1,1}$ ($A \cup s_{1,2}$ respectively). Consider now $A \cup s_{1,1} \cup s_{1,2}$ and s_2 s.t. s_2 is small in $A \cup s_{1,1} \cup s_{1,2} \cup s_2$. Continue with $s_{3,1}$, $s_{3,2}$ small in $A \cup s_{1,1} \cup s_{1,2} \cup s_2 \cup s_{3,1}$ etc.

Without additional properties, this system creates a new rule, which is not equivalent to any usual rules.

□

3.2 The finite versions

Fact 3.6

(+++ Orig. No.: Fact I-n +++)

LABEL: Fact I-n

(1) (I_n) + $(eM\mathcal{I})$ $\Rightarrow$ $(\mathcal{M}_n^+)$,

(2) (I_n) + $(eM\mathcal{I})$ $\Rightarrow$ (CM_n),

(3) (I_n) + $(eM\mathcal{I})$ $\Rightarrow$ (OR_n).

Proof

(+++ Orig.: Proof +++)

(1)

Let $X_1 \subseteq \ldots \subseteq X_n$, so $X_n = X_1 \cup (X_2 - X_1) \cup \ldots \cup (X_n - X_{n-1})$. Let $X_i \in \mathcal{F}(X_{i+1})$, so $X_{i+1} - X_i \in \mathcal{I}(X_{i+1}) \subseteq \mathcal{I}(X_n)$ by $(eM\mathcal{I})$ for $1 \leq i \leq n-1$, so by (I_n) $X_1 \in \mathcal{M}^+(X_n)$.

(2)

Suppose $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_{n-1}$, but $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \mid\sim \neg\beta_{n-1}$. Then $M(\alpha \wedge \neg\beta_1), \ldots, M(\alpha \wedge \neg\beta_{n-1}) \in \mathcal{I}(M(\alpha))$, and $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \wedge \beta_{n-1}) \in \mathcal{I}(M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2})) \subseteq \mathcal{I}(M(\alpha))$ by $(eM\mathcal{I})$. But $M(\alpha) = M(\alpha \wedge \neg\beta_1) \cup \ldots \cup M(\alpha \wedge \neg\beta_{n-1}) \cup M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \wedge \beta_{n-1})$ is now the union of n small subsets, *contradiction.*

(3)

Let $\alpha_1 \mid\sim \beta, \ldots, \alpha_{n-1} \mid\sim \beta$, so $M(\alpha_i \wedge \neg\beta) \in \mathcal{I}(M(\alpha_i))$ for $1 \leq i \leq n-1$, so $M(\alpha_i \wedge \neg\beta) \in \mathcal{I}(M(\alpha_1 \vee \ldots \vee \alpha_{n-1}))$ for $1 \leq i \leq n-1$ by $(eM\mathcal{I})$, so $M((\alpha_1 \vee \ldots \vee \alpha_{n-1}) \wedge \beta) = M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) - \cup\{M(\alpha_i \wedge \neg\beta) : 1 \leq i \leq n-1\} \notin \mathcal{I}(M(\alpha_1 \vee \ldots \vee \alpha_{n-1}))$ by (I_n), so $\alpha_1 \vee \ldots \vee \alpha_{n-1} \not\mid\sim \neg\beta$.

□

In the following example, (OR_n), $(\mathcal{M}_n^+)$, (CM_n) hold, but $(\mathcal{I}_n)$ fails, so by Fact 3.6 (page 7) $(\mathcal{I}_n)$ is strictly stronger than (OR_n), $(\mathcal{M}_n^+)$, (CM_n).

Example 3.1

(+++ Orig. No.: Example Not-I-n +++)

LABEL: Example Not-I-n

Let $n \geq 3$.

Consider $X := \{1, \ldots, n\}$, $\mathcal{Y} := \mathcal{P}(X) - \{\emptyset\}$, $\mathcal{I}(X) := \{\emptyset\} \cup \{\{i\} : 1 \leq i \leq n\}$, and for all $Y \subset X$ $\mathcal{I}(Y) := \{\emptyset\}$.

(Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ (by Fact 3.1 (page 6)), $(1 * s)$, $(2 * s)$ hold, (I_n) fails, of course.

(1) (OR_n) holds:

Suppose $\alpha_1 \mid\sim \beta, \ldots, \alpha_{n-1} \mid\sim \beta$, $\alpha_1 \vee \ldots \vee \alpha_{n-1} \mid\sim \neg\beta$.

Case 1: $\alpha_1 \vee \ldots \vee \alpha_{n-1} \vdash \neg\beta$, then for all i $\alpha_i \vdash \neg\beta$, so for no i $\alpha_i \mid\sim \beta$ by $(1 * s)$ and thus (AND_1), *contradiction.*

Case 2: $\alpha_1 \vee \ldots \vee \alpha_{n-1} \not\vdash \neg\beta$, then $M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) = X$, and there is exactly 1 $k \in X$ s.t. $k \models \beta$. Fix this k. By prerequisite, $\alpha_i \mid\sim \beta$. If $M(\alpha_i) = X$, $\alpha_i \vdash \beta$ cannot be, so there must be exactly 1 k' s.t. $k' \models \neg\beta$, but $card(X) \geq 3$, *contradiction.* So $M(\alpha_i) \subset X$, and $\alpha_i \vdash \beta$, so $M(\alpha_i) = \emptyset$ or $M(\alpha_i) = \{k\}$ for all i, so $M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) \neq X$, *contradiction.*

(2) $(\mathcal{M}^+_n)$ holds:

$(\mathcal{M}^+_n)$ is a consequence of $(\mathcal{M}^+_\omega)$, (3) so it suffices to show that the latter holds. Let $X_1 \in \mathcal{F}(X_2)$, $X_2 \in \mathcal{F}(X_3)$. Then $X_1 = X_2$ or $X_2 = X_3$, so the result is trivial.

(3) (CM_n) holds:

Suppose $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_{n-1}$, $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \mid\sim \neg\beta_{n-1}$.

Case 1: For all i, $1 \leq i \leq n-2$, $\alpha \vdash \beta_i$, then $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2}) = M(\alpha)$, so $\alpha \mid\sim \beta_{n-1}$ and $\alpha \mid\sim \neg\beta_{n-1}$, *contradiction.*

Case 2: There is i, $1 \leq i \leq n-2$, $\alpha \not\vdash \beta_i$, then $M(\alpha) = X$, $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2}) \subset M(\alpha)$, so $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \vdash \neg\beta_{n-1}$. $Card(M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2})) \geq n - (n-2) = 2$, so $card(M(\neg\beta_{n-1})) \geq 2$, so $\alpha \not\mid\sim \beta_{n-1}$, *contradiction.*

□

3.3 The ω version

Fact 3.7

(+++ Orig. No.: Fact CM-Omega +++)

LABEL: Fact CM-Omega

$(CM_\omega) \Leftrightarrow (\mathcal{M}^+_\omega)$ (4)

Proof

(+++ Orig.: Proof +++)

" $\Rightarrow$ "

Suppose all sets are definable.

Let $A, B \in \mathcal{I}(X)$, $X = M(\alpha)$, $A = M(\alpha \wedge \neg\beta)$, $B = M(\alpha \wedge \neg\beta')$, so $\alpha \mid\sim \beta$, $\alpha \mid\sim \beta'$, so by (CM_ω) $\alpha \wedge \beta' \mid\sim \beta$, so $A - B = M(\alpha \wedge \beta' \wedge \neg\beta) \in \mathcal{I}(M(\alpha \wedge \beta')) = \mathcal{I}(X{-}\mathrm{B})$.

" $\Leftarrow$ "

Let $\alpha \mid\sim \beta$, $\alpha \mid\sim \beta'$, so $M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha))$, $M(\alpha \wedge \neg\beta') \in \mathcal{I}(M(\alpha))$, so by prerequisite $M(\alpha \wedge \neg\beta') - M(\alpha \wedge \neg\beta) = M(\alpha \wedge \beta \wedge \neg\beta') \in \mathcal{I}(M(\alpha) - M(\alpha \wedge \neg\beta)) = \mathcal{I}(M(\alpha \wedge \beta))$, so $\alpha \wedge \beta \mid\sim \beta'$.

□

Fact 3.8

(+++ Orig. No.: Fact I-Omega +++)

LABEL: Fact I-Omega

(1) (I_ω) + $(eM\mathcal{I}) \Rightarrow (OR_\omega)$,

(2) (I_ω) + $(eM\mathcal{I}) \Rightarrow (\mathcal{M}^+_\omega)$ (1),

(3) (I_ω) + $(eM\mathcal{F}) \Rightarrow (\mathcal{M}^+_\omega)$ (2),

(4) (I_ω) + $(eM\mathcal{I}) \Rightarrow (\mathcal{M}^+_\omega)$ (3),

(5) (I_ω) + $(eM\mathcal{F}) \Rightarrow (\mathcal{M}^+_\omega)$ (4) (and thus, by Fact 3.7 (page 8), (CM_ω)).

Proof

(+++ Orig.: Proof +++)

(1)

Let $\alpha \mid\!\sim \beta$, $\alpha' \mid\!\sim \beta \Rightarrow M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha))$, $M(\alpha' \wedge \neg\beta) \in \mathcal{I}(M(\alpha'))$, so by $(eM\mathcal{I})$ $M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$, $M(\alpha' \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$, so $M((\alpha \vee \alpha') \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$ by (I_ω), so $\alpha \vee \alpha' \mid\!\sim \beta$.

(2)

Let $A \subseteq X \subseteq Y$, $A \in \mathcal{I}(Y)$, $X - A \in \mathcal{I}(X) \subseteq_{(eM\mathcal{I})} \mathcal{I}(Y) \Rightarrow X = (X - A) \cup A \in \mathcal{I}(Y)$ by (I_ω).

(3)

Let $A \subseteq X \subseteq Y$, let $A \in \mathcal{I}(Y)$, $Y - X \in \mathcal{I}(Y) \Rightarrow A \cup (Y - X) \in \mathcal{I}(Y)$ by $(I_\omega) \Rightarrow X - A = Y - (A \cup (Y - X)) \in \mathcal{F}(Y) \Rightarrow X - A \in \mathcal{F}(X)$ by $(eM\mathcal{F})$.

(4)

Let $A \subseteq X \subseteq Y$, $A \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y)$, so $Y - X \in \mathcal{I}(Y)$, $X - A \in \mathcal{I}(X) \subseteq_{(eM\mathcal{I})} \mathcal{I}(Y) \Rightarrow Y - A = (Y - X) \cup (X - A) \in \mathcal{I}(Y)$ by $(\mathcal{I}_\omega) \Rightarrow A \in \mathcal{F}(Y)$.

(5)

Let $A, B \subseteq X$, $A, B \in \mathcal{I}(X) \Rightarrow_{(I_\omega)} A \cup B \in \mathcal{I}(X) \Rightarrow X - (A \cup B) \in \mathcal{F}(X)$, but $X - (A \cup B) \subseteq X - \mathrm{B}$, so $X - (A \cup B) \in \mathcal{F}(X - \mathrm{B})$ by $(eM\mathcal{F})$, so $A - B = (X - B) - (X - (A \cup B)) \in \mathcal{I}(X - \mathrm{B})$.

□

We give three examples of independence of the various versions of $(\mathcal{M}^+_\omega)$.

Example 3.2

(+++ Orig. No.: Example Versions-M-Omega +++)

LABEL: Example Versions-M-Omega

All numbers refer to the versions of $(\mathcal{M}^+_\omega)$.

For easier reading, we re-write for $A \subseteq X \subseteq Y$

$(\mathcal{M}^+_\omega)(1) : A \in \mathcal{F}(X),\ A \in \mathcal{I}(Y) \Rightarrow X \in \mathcal{I}(Y)$,

$(\mathcal{M}^+_\omega)(2) : X \in \mathcal{F}(Y),\ A \in \mathcal{I}(Y) \Rightarrow A \in \mathcal{I}(X)$.

We give three examples. Investigating all possibilities exhaustively seems quite tedious, and might best be done with the help of a computer. Fact 3.1 (page 6) will be used repeatedly.

- (1), (2), (4) fail, (3) holds:
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, c\}, \{b, c\}, Y\}$
  Let $X := \{a, b\}$, $\mathcal{F}(X) := \{\{a\}, X\}$, $A := \{a\}$, and $\mathcal{F}(Z) := \{Z\}$ for all $Z \neq X, Y$.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ hold, (I_ω) fails, of course.
  (1) fails: $A \in \mathcal{F}(X)$, $A \in \mathcal{I}(Y)$, $X \notin \mathcal{I}(Y)$.
  (2) fails: $\{a, c\} \in \mathcal{F}(Y)$, $\{a\} \in \mathcal{I}(Y)$, but $\{a\} \notin \mathcal{I}(\{a, c\})$.
  (3) holds: If $X_1 \in \mathcal{F}(X_2)$, $X_2 \in \mathcal{F}(X_3)$, then $X_1 = X_2$ or $X_2 = X_3$, so (3) holds trivially (note that $X \notin \mathcal{F}(Y)$).
  (4) fails: $\{a\}, \{b\} \in \mathcal{I}(Y)$, $\{a\} \notin \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\}) = \{\emptyset\}$.
- (2), (3), (4) fail, (1) holds:
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, b\}, \{a, c\}, Y\}$
  Let $X := \{a, b\}$, $\mathcal{F}(X) := \{\{a\}, X\}$, and $\mathcal{F}(Z) := \{Z\}$ for all $Z \neq X, Y$.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ hold, (I_ω) fails, of course.
  (1) holds:
  Let $X_1 \in \mathcal{F}(X_2)$, $X_1 \in \mathcal{I}(X_3)$, we have to show $X_2 \in \mathcal{I}(X_3)$. If $X_1 = X_2$, then this is trivial. Consider $X_1 \in \mathcal{F}(X_2)$. If $X_1 \neq X_2$, then X_1 has to be $\{a\}$ or $\{a, b\}$ or $\{a, c\}$. But none of these are in $\mathcal{I}(X_3)$ for any X_3, so the implication is trivially true.
  (2) fails: $\{a, c\} \in \mathcal{F}(Y)$, $\{c\} \in \mathcal{I}(Y)$, $\{c\} \notin \mathcal{I}(\{a, c\})$.
  (3) fails: $\{a\} \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y)$, $\{a\} \notin \mathcal{F}(Y)$.
  (4) fails: $\{b\}, \{c\} \in \mathcal{I}(Y)$, $\{c\} \notin \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\}) = \{\emptyset\}$.
- (1), (2), (4) hold, (3) fails:
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, b\}, \{a, c\}, Y\}$
  Let $\mathcal{F}(\{a, b\}) := \{\{a\}, \{a, b\}\}$, $\mathcal{F}(\{a, c\}) := \{\{a\}, \{a, c\}\}$, and $\mathcal{F}(Z) := \{Z\}$ for all other Z.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ hold, (I_ω) fails, of course.
  (1) holds:
  Let $X_1 \in \mathcal{F}(X_2)$, $X_1 \in \mathcal{I}(X_3)$, we have to show $X_2 \in \mathcal{I}(X_3)$. Consider $X_1 \in \mathcal{I}(X_3)$. If $X_1 = X_2$, this is trivial. If $\emptyset \neq X_1 \in \mathcal{I}(X_3)$, then $X_1 = \{b\}$ or $X_1 = \{c\}$, but then by $X_1 \in \mathcal{F}(X_2)$ X_2 has to be $\{b\}$, or $\{c\}$, so $X_1 = X_2$.

(2) holds: Let $X_1 \subseteq X_2 \subseteq X_3$, let $X_2 \in \mathcal{F}(X_3)$, $X_1 \in \mathcal{I}(X_3)$, we have to show $X_1 \in \mathcal{I}(X_2)$. If $X_1 = \emptyset$, this is trivial, likewise if $X_2 = X_3$. Otherwise $X_1 = \{b\}$ or $X_1 = \{c\}$, and $X_3 = Y$. If $X_1 = \{b\}$, then $X_2 = \{a, b\}$, and the condition holds, likewise if $X_1 = \{c\}$, then $X_2 = \{a, c\}$, and it holds again.

(3) fails: $\{a\} \in \mathcal{F}(\{a, c\})$, $\{a, c\} \in \mathcal{F}(Y)$, $\{a\} \notin \mathcal{F}(Y)$.

(4) holds:

If $A, B \in \mathcal{I}(X)$, and $A \neq B$, $A, B \neq \emptyset$, then $X = Y$ and e.g. $A = \{c\}$, $B = \{b\}$, and $\{c\} \in \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\})$.

□

3.4 Rational Monotony

Fact 3.9

(+++ Orig. No.: Fact M-plus-plus +++)

LABEL: Fact M-plus-plus

The three versions of $(\mathcal{M}^{++})$ are equivalent.

(We assume closure of the domain under set difference. For the third version of $(\mathcal{M}^{++})$, we use (iM).)

Proof

(+++ Orig.: Proof +++)

For (1) and (2), we have $A, B \subseteq X$, for (3) we have $A \subseteq X \subseteq Y$. For $A, B \subseteq X$, $(X - B) - ((X - A) - B) = A - B$ holds.

(1) $\Rightarrow$ (2) : Let $A \in \mathcal{F}(X)$, $B \notin \mathcal{F}(X)$, so $X - A \in \mathcal{I}(X)$, so by prerequisite $(X - A) - B \in \mathcal{I}(X\text{−B})$, so $A - B = (X - B) - ((X - A) - B) \in \mathcal{F}(X\text{−B})$.

(2) $\Rightarrow$ (1) : Let $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X)$, so $X - A \in \mathcal{F}(X)$, so by prerequisite $(X - A) - B \in \mathcal{F}(X\text{−B})$, so $A - B = (X - B) - ((X - A) - B) \in \mathcal{I}(X\text{−B})$.

(1) $\Rightarrow$ (3) :

Suppose $A \notin \mathcal{M}^+(Y)$, but $X \in \mathcal{M}^+(Y)$, we show $A \notin \mathcal{M}^+(X)$. So $A \in \mathcal{I}(Y)$, $Y - X \notin \mathcal{F}(Y)$, so by (1) $A = A - (Y - X) \in \mathcal{I}(Y - (Y - X)) = \mathcal{I}(X)$.

(3) $\Rightarrow$ (1) :

Suppose $A - B \notin \mathcal{I}(X\text{−B})$, $B \notin \mathcal{F}(X)$, we show $A \notin \mathcal{I}(X)$. By prerequisite $A - B \in \mathcal{M}^+(X\text{−B})$, $X - B \in \mathcal{M}^+(X)$, so by (3) $A - B \in \mathcal{M}^+(X)$, so by (iM) $A \in \mathcal{M}^+(X)$, so $A \notin \mathcal{I}(X)$.

□

Fact 3.10

(+++ Orig. No.: Fact M-RatM +++)

LABEL: Fact M-RatM

We assume that all sets are definable by a formula.

$(RatM) \Leftrightarrow (\mathcal{M}^{++})$

Proof

(+++ Orig.: Proof +++)

We show equivalence of $(RatM)$ with version (1) of $(\mathcal{M}^{++})$.

" $\Rightarrow$ "

We have $A, B \subseteq X$, so we can write $X = M(\phi)$, $A = M(\phi \wedge \neg\psi)$, $B = M(\phi \wedge \neg\psi')$. $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X)$, so $\phi \mid\sim \psi$, $\phi \not\mid\sim \neg\psi'$, so by $(RatM)$ $\phi \wedge \psi' \mid\sim \psi$, so $A - B = M(\phi \wedge \neg\psi) - M(\phi \wedge \neg\psi') = M(\phi \wedge \psi' \wedge \neg\psi) \in \mathcal{I}(M(\phi \wedge \psi')) = \mathcal{I}(X\text{−B})$.

" $\Leftarrow$ "

Let $\phi \mid\sim \psi$, $\phi \not\mid\sim \neg\psi'$, so $M(\phi \wedge \neg\psi) \in \mathcal{I}(M(\phi))$, $M(\phi \wedge \neg\psi') \notin \mathcal{F}(M(\phi))$, so by $(\mathcal{M}^{++})$ (1) $M(\phi \wedge \psi' \wedge \neg\psi) = M(\phi \wedge \neg\psi) - M(\phi \wedge \neg\psi') \in \mathcal{I}(M(\phi \wedge \psi'))$, so $\phi \wedge \psi' \mid\sim \psi$.

□

4 Size and principal filter logic

LABEL: Section Principal

The connection with logical rules is shown in the following table Definition 4 (page 11). Most of the table was already published in [GS08c], it is repeated here for the reader's convenience.

LABEL: Definition Log-Cond-Ref-Size

The numbers in the first column "Correspondence" refer to Proposition 21 in [GS08c], those in the second column "Correspondence" to Proposition 4.1 (page 12).

Logical rule		Correspondence	Model set	Correspondence	Size Rules
		Basics			
(SC) Supraclassicality $\phi \vdash \psi \Rightarrow \phi \mid\sim \psi$ (REF) Reflexivity $T \cup \{\alpha\} \mid\sim \alpha$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ (4.1) $\Leftarrow$ (4.2)	$(\mu \subseteq)$ $f(X) \subseteq X$	trivial	(Opt)
(LLE) Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \mid\sim \psi \Rightarrow$ $\phi' \mid\sim \psi$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$				
(RW) Right Weakening $\phi \mid\sim \psi, \vdash \psi \to \psi' \Rightarrow$ $\phi \mid\sim \psi'$	(RW) $T \mid\sim \psi, \vdash \psi \to \psi' \Rightarrow$ $T \mid\sim \psi'$			trivial	(iM)
(wOR) $\phi \mid\sim \psi$, $\phi' \vdash \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (3.1) $\Leftarrow$ (3.2)	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$	$\Leftarrow$ (1.1) $\Rightarrow$ (1.2)	$(eM\mathcal{I})$
$(disjOR)$ $\phi \vdash \neg \phi'$, $\phi \mid\sim \psi$, $\phi' \mid\sim \psi \Rightarrow \phi \vee \phi' \mid\sim \psi$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (2.1) $\Leftarrow$ (2.2)	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftarrow$ (4.1) $\Rightarrow$ (4.2)	$(I \cup disj)$
(CP) Consistency Preservation $\phi \mid\sim \bot \Rightarrow \phi \vdash \bot$	(CP) $T \mid\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ (5.1) $\Leftarrow$ (5.2)	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I_1)
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I_1)
	(AND_1) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$				(I_2)
	(AND_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_{n-1} \Rightarrow$ $\alpha \not\mid\sim (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$				(I_n)
(AND) $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow$ $\phi \mid\sim \psi \wedge \psi'$	(AND) $T \mid\sim \psi, T \mid\sim \psi' \Rightarrow$ $T \mid\sim \psi \wedge \psi'$			trivial	(I_ω)
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed			trivial	$(iM) + (I_\omega)$
(OR) $\phi \mid\sim \psi, \phi' \mid\sim \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (1.1) $\Leftarrow$ (1.2)	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftarrow$ (2.1) $\Rightarrow$ (2.2)	$(eM\mathcal{I}) + (I_\omega)$
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	$\Rightarrow$ (6.1) $\Leftarrow (\mu dp) + (\mu \subseteq)$ (6.2) $\not\Leftarrow$ without (μdp) (6.3) $\Leftarrow (\mu \subseteq)$ (6.4) T' a formula $\Leftarrow$ (6.5) T' a formula	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$ $(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$	$\Leftarrow$ (3.1) $\Rightarrow$ (3.2)	$(eM\mathcal{I}) + (I_\omega)$
(CUT) $T \mid\sim \alpha; T \cup \{\alpha\} \mid\sim \beta \Rightarrow$ $T \mid\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ (7.1) $\Leftarrow$ (7.2)	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$	$\Leftarrow$ (8.1) $\not\Rightarrow$ (8.2)	$(eM\mathcal{I}) + (I_\omega)$

Logical rule		Correspondence	Model set	Correspondence	Size-Rule
			Cumulativity		
(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow \alpha' \mid\sim \beta$				trivial	$(eM\mathcal{F})$
(CM_2) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow \alpha \wedge \beta \not\vdash \neg \beta'$					(I_2)
(CM_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \not\vdash \neg \beta_n$					(I_n)
(CM) Cautious Monotony $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow$ $\phi \wedge \psi \mid\sim \psi'$	(CM) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ (8.1) $\Leftarrow$ (8.2)	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	$\Leftarrow$ (5.1) $\Rightarrow$ (5.2)	$(\mathcal{M}^+_\omega)(4)$
or $(ResM)$ Restricted Monotony $T \mid\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\sim \beta$		$\Rightarrow$ (9.1) $\Leftarrow$ (9.2)	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\phi \mid\sim \psi \Rightarrow$ $(\phi \mid\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\sim \psi')$	(CUM) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ (11.1) $\Leftarrow$ (11.2)	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	$\Leftarrow$ (9.1) $\not\Rightarrow$ (9.2)	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}^+_\omega)(4)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ (10.1) $\Leftarrow$ (10.2)	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	$\Leftarrow$ (10.1) $\not\Rightarrow$ (10.2)	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$
			Rationality		
$(RatM)$ Rational Monotony $\phi \mid\sim \psi, \phi \not\mid\sim \neg \psi' \Rightarrow$ $\phi \wedge \psi' \mid\sim \psi$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (12.1) $\Leftarrow$ (μdp) (12.2) $\not\Leftarrow$ without (μdp) (12.3) $\Leftarrow$ T a formula (12.4)	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	$\Leftarrow$ (6.1) $\Rightarrow$ (6.2)	$(\mathcal{M}^{++})$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (13.1) $\Leftarrow$ (μdp) (13.2) $\not\Leftarrow$ without (μdp) (13.3) $\Leftarrow$ T a formula (13.4)	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (14.1) $\Leftarrow$ (μdp) (14.2) $\not\Leftarrow$ without (μdp) (14.3) $\Leftarrow$ T a formula (14.4)	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ (15.1) $\Leftarrow$ (15.2)	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$		
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow$ $(\mu \subseteq) + (\mu =)$ (16.1) $\Leftarrow$ (μdp) (16.2) $\not\Leftarrow$ without (μdp) (16.3)	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow$ $(\mu \subseteq) + (\mu =)$ (17.1) $\Leftarrow$ (μdp) (17.2) $\not\Leftarrow$ without (μdp) (17.3)	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$		

(1) to (7) of the following proposition (in different notation, as the more systematic connections were found only afterwards) was already published in [GS08c], we give it here in totality to complete the picture.

Proposition 4.1

(+++ Orig. No.: Proposition Ref-Class-Mu-neu +++)

LABEL: Proposition Ref-Class-Mu-neu

If $f(X)$ is the smallest A s.t. $A \in \mathcal{F}(X)$, then, given the property on the left, the one on the right follows.

Conversely, when we define $\mathcal{F}(X) := \{X' : f(X) \subseteq X' \subseteq X\}$, given the property on the right, the one on the left follows. For this direction, we assume that we can use the full powerset of some base set U - as is the case for the model sets of a finite language. This is perhaps not too bold, as we mainly want to stress here the intuitive connections, without putting too much weight on definability questions.

We assume (iM) to hold.

(1.1)	$(eM\mathcal{I})$	$\Rightarrow$	(μwOR)
(1.2)		$\Leftarrow$	
(2.1)	$(eM\mathcal{I}) + (I_\omega)$	$\Rightarrow$	(μOR)
(2.2)		$\Leftarrow$	
(3.1)	$(eM\mathcal{I}) + (I_\omega)$	$\Rightarrow$	(μPR)
(3.2)		$\Leftarrow$	
(4.1)	$(I \cup disj)$	$\Rightarrow$	$(\mu disjOR)$
(4.2)		$\Leftarrow$	
(5.1)	$(\mathcal{M}^+_\omega)(4)$	$\Rightarrow$	(μCM)
(5.2)		$\Leftarrow$	
(6.1)	$(\mathcal{M}^{++})$	$\Rightarrow$	$(\mu RatM)$
(6.2)		$\Leftarrow$	
(7.1)	(I_ω)	$\Rightarrow$	(μAND)
(7.2)		$\Leftarrow$	
(8.1)	$(eM\mathcal{I}) + (I_\omega)$	$\Rightarrow$	(μCUT)
(8.2)		$\not\Leftarrow$	
(9.1)	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}^+_\omega)(4)$	$\Rightarrow$	(μCUM)
(9.2)		$\not\Leftarrow$	
(10.1)	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(10.2)		$\not\Leftarrow$	

Note that there is no (μwCM), as the conditions $(\mu \ldots)$ imply that the filter is principal, and thus that (I_ω) holds - we cannot "see" (wCM) alone with principal filters.

Proof

(+++ Orig.: Proof +++)

(1.1) $(eM\mathcal{I}) \Rightarrow (\mu wOR)$:

$X - f(X)$ is small in X, so it is small in $X \cup Y$ by $(eM\mathcal{I})$, so $A := X \cup Y - (X - f(X)) \in \mathcal{F}(X \cup Y)$, but $A \subseteq f(X) \cup Y$, and $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, so $f(X \cup Y) \subseteq A \subseteq f(X) \cup Y$.

(1.2) $(\mu wOR) \Rightarrow (eM\mathcal{I})$:

Let $X \subseteq Y$, $X' := Y{-}X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X{-}A$, so $f(X \cup X') \subseteq f(X) \cup X' \subseteq (X - A) \cup X'$ by prerequisite, so $(X \cup X') - ((X - A) \cup X') = A \in \mathcal{I}(X \cup X')$.

(2.1) $(eM\mathcal{I}) + (I_\omega) \Rightarrow (\mu OR)$:

$X - f(X)$ is small in X, $Y - f(Y)$ is small in Y, so both are small in $X \cup Y$ by $(eM\mathcal{I})$, so $A := (X - f(X)) \cup (Y - f(Y))$ is small in $X \cup Y$ by (I_ω), but $X \cup Y - (f(X) \cup f(Y)) \subseteq A$, so $f(X) \cup f(Y) \in \mathcal{F}(X \cup Y)$, so, as $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, $f(X \cup Y) \subseteq f(X) \cup f(Y)$.

(2.2) $(\mu OR) \Rightarrow (eM\mathcal{I}) + (I_\omega)$:

Let again $X \subseteq Y$, $X' := Y{-}X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X{-}A$. $f(X') \subseteq X'$, so $f(X \cup X') \subseteq f(X) \cup f(X') \subseteq (X - A) \cup X'$ by prerequisite, so $(X \cup X') - ((X - A) \cup X') = A \in \mathcal{I}(X \cup X')$.

(I_ω) holds by definition.

(3.1) $(eM\mathcal{I}) + (I_\omega) \Rightarrow (\mu PR)$:

Let $X \subseteq Y$. $Y - f(Y)$ is the largest element of $\mathcal{I}(Y)$, $X - f(X) \in \mathcal{I}(X) \subseteq \mathcal{I}(Y)$ by $(eM\mathcal{I})$, so $(X - f(X)) \cup (Y - f(Y)) \in \mathcal{I}(Y)$ by (I_ω), so by "largest" $X - f(X) \subseteq Y - f(Y)$, so $f(Y) \cap X \subseteq f(X)$.

(3.2) $(\mu PR) \Rightarrow (eM\mathcal{I}) + (I_\omega)$

Let again $X \subseteq Y$, $X' := Y{-}X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X{-}A$, so by prerequisite $f(Y) \cap X \subseteq X{-}A$, so $f(Y) \subseteq X' \cup (X{-}A)$, so $(X \cup X') - (X' \cup (X - A)) = A \in \mathcal{I}(Y)$.

Again, (I_ω) holds by definition.

(4.1) $(I \cup disj) \Rightarrow (\mu disjOR)$:

If $X \cap Y = \emptyset$, then (1) $A \in \mathcal{I}(X), B \in \mathcal{I}(Y) \Rightarrow A \cup B \in \mathcal{I}(X \cup Y)$ and (2) $A \in \mathcal{F}(X), B \in \mathcal{F}(Y) \Rightarrow A \cup B \in \mathcal{F}(X \cup Y)$ are equivalent. (By $X \cap Y = \emptyset$, $(X - A) \cup (Y - B) = (X \cup Y) - (A \cup B)$.) So $f(X) \in \mathcal{F}(X)$, $f(Y) \in \mathcal{F}(Y) \Rightarrow$ (by prerequisite) $f(X) \cup f(Y) \in \mathcal{F}(X \cup Y)$. $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, so $f(X \cup Y) \subseteq f(X) \cup f(Y)$.

(4.2) $(\mu disjOR) \Rightarrow (I \cup disj)$:

Let $X \subseteq Y$, $X' := Y{-}X$. Let $A \in \mathcal{I}(X)$, $A' \in \mathcal{I}(X')$, so $X - A \in \mathcal{F}(X)$, $X' - A' \in \mathcal{F}(X')$, so $f(X) \subseteq X{-}A$, $f(X') \subseteq X' - A'$, so $f(X \cup X') \subseteq f(X) \cup f(X') \subseteq (X - A) \cup (X' - A')$ by prerequisite, so $(X \cup X') - ((X - A) \cup (X' - A')) = A \cup A' \in \mathcal{I}(X \cup X')$.

(5.1) $(\mathcal{M}^+_\omega) \Rightarrow (\mu CM)$:

$f(X) \subseteq Y \subseteq X \Rightarrow X - Y \in \mathcal{I}(X)$, $X - f(X) \in \mathcal{I}(X) \Rightarrow$ (by $(\mathcal{M}^+_\omega)$, (4)) $A := (X - f(X)) - (X - Y) \in \mathcal{I}(Y) \Rightarrow Y - A = f(X) - (X - Y) \in \mathcal{F}(Y) \Rightarrow f(Y) \subseteq f(X) - (X - Y) \subseteq f(X)$.

(5.2) $(\mu CM) \Rightarrow (\mathcal{M}^+_\omega)$

Let $X - A \in \mathcal{I}(X)$, so $A \in \mathcal{F}(X)$, let $B \in \mathcal{I}(X)$, so $f(X) \subseteq X - B \subseteq X$, so by prerequisite $f(X - B) \subseteq f(X)$. As $A \in \mathcal{F}(X)$, $f(X) \subseteq A$, so $f(X - B) \subseteq f(X) \subseteq A \cap (X - B) = A{-}B$, and $A - B \in \mathcal{F}(X{-}B)$, so $(X - A) - B = X - (A \cup B) =$

$(X-B)-(A-B) \in \mathcal{I}(X-$B), so $(\mathcal{M}^+_\omega)$, (4) holds.

(6.1) $(\mathcal{M}^{++}) \Rightarrow (\mu RatM):$

Let $X \subseteq Y$, $X \cap f(Y) \neq \emptyset$. If $Y-X \in \mathcal{F}(Y)$, then $A := (Y-X) \cap f(Y) \in \mathcal{F}(Y)$, but by $X \cap f(Y) \neq \emptyset$ $A \subset f(Y)$, contradicting "smallest" of $f(Y)$. So $Y-X \notin \mathcal{F}(Y)$, and by $(\mathcal{M}^{++})$ $X - f(Y) = (Y - f(Y)) - (Y-X) \in \mathcal{I}(X)$, so $X \cap f(Y) \in \mathcal{F}(X)$, so $f(X) \subseteq f(Y) \cap X$.

(6.2) $(\mu RatM) \Rightarrow (\mathcal{M}^{++})$

Let $A \in \mathcal{F}(Y)$, $B \notin \mathcal{F}(Y)$. $B \notin \mathcal{F}(Y) \Rightarrow Y-B \notin \mathcal{I}(Y) \Rightarrow (Y-B) \cap f(Y) \neq \emptyset$. Set $X := Y-$B, so $X \cap f(Y) \neq \emptyset$, $X \subseteq Y$, so $f(X) \subseteq f(Y) \cap X$ by prerequisite. $f(Y) \subseteq A \Rightarrow f(X) \subseteq f(Y) \cap X = f(Y) - B \subseteq A-$B.

(7.1) $(\mathcal{I}_\omega) \Rightarrow (\mu AND)$

Trivial.

(7.2) $(\mu AND) \Rightarrow (\mathcal{I}_\omega)$

Trivial.

(8.1) Let $f(X) \subseteq Y \subseteq X$. $Y - f(Y) \in \mathcal{I}(Y) \subseteq \mathcal{I}(X)$ by $(eM\mathcal{I})$. $f(X) \subseteq Y \Rightarrow X-Y \subseteq X - f(X) \in \mathcal{I}(X)$, so by (iM) $X-Y \in \mathcal{I}(X)$. Thus by (I_ω) $X - f(Y) = (X-Y) \cup (Y - f(Y)) \in \mathcal{I}(X)$, so $f(Y) \in \mathcal{F}(X)$, so $f(X) \subseteq f(Y)$ by definition.

(8.2) (μCUT) is too special to allow to deduce $(eM\mathcal{I})$. Consider $U := \{a,b,c\}$, $X := \{a,b\}$, $\mathcal{F}(X) = \{X, \{a\}\}$, $\mathcal{F}(Z) = \{Z\}$ for all other $X \neq Z \subseteq U$. Then $(eM\mathcal{I})$ fails, as $\{b\} \in \mathcal{I}(X)$, but $\{b\} \notin \mathcal{I}(U)$. (iM) and $(eM\mathcal{F})$ hold. We have to check $f(A) \subseteq B \subseteq A \Rightarrow f(A) \subseteq f(B)$. The only case where it might fail is $A = X$, $B = \{a\}$, but it holds there, too.

(9.1) By Fact 14 in [GS08c], (6), we have $(\mu CM) + (\mu CUT) \Leftrightarrow (\mu CUM)$, so the result follows from (5.1) and (8.1).

(9.2) Consider the same example as in (8.2). $f(A) \subseteq B \subseteq A \Rightarrow f(A) = f(B)$ holds there, too, by the same argument as above.

(10.1) Let $f(X) \subseteq Y$, $f(Y) \subseteq X$. So $f(X), f(Y) \subseteq X \cap Y$, and $X - (X \cap Y) \in \mathcal{I}(X)$, $Y - (X \cap Y) \in \mathcal{I}(Y)$ by (iM). Thus $f(X), f(Y) \in \mathcal{F}(X \cap Y)$ by $(eM\mathcal{F})$ and $f(X) \cap f(Y) \in \mathcal{F}(X \cap Y)$ by (I_ω). So $X \cap Y - (f(X) \cap f(Y)) \in \mathcal{I}(X \cap Y)$, so $X \cap Y - (f(X) \cap f(Y)) \in \mathcal{I}(X), \mathcal{I}(Y)$ by $(eM\mathcal{I})$, so $(X - (X \cap Y)) \cup (X \cap Y - f(X) \cap f(Y)) = X - f(X) \cap f(Y) \in \mathcal{I}(X)$ by (I_ω), so $f(X) \cap f(Y) \in \mathcal{F}(X)$, likewise $f(X) \cap f(Y) \in \mathcal{F}(Y)$, so $f(X) \subseteq f(X) \cap f(Y)$, $f(Y) \subseteq f(X) \cap f(Y)$, and $f(X) = f(Y)$.

(10.2) Consider again the same example as in (8.2), we have to show that $f(A) \subseteq B$, $f(B) \subseteq A \Rightarrow f(A) = f(B)$. The only interesting case is when one of A, B is X, but not both. Let e.g. $A = X$. We then have $f(X) = \{a\}$, $f(B) = B \subseteq X$, and $f(X) = \{a\} \subseteq B$, so $B = \{a\}$, and the condition holds.

□

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